The creeping motion of a non-Newtonian fluid past a sphere

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The equations of motion and continuity are solved together with the slow-flow stress equations for an incompressible Rivlin-Ericksen fluid. The boundary conditions for slow flow past a sphere are satisfied by matching inner (Stokes) and outer (Oseen) Reynolds-number expansions of the stream function. The terms in the inner expansion are the solutions of non-linear partial differential equations which are solved approximately by expanding in terms of a non-Newtonian parameter λ. The drag force on the sphere is obtained from the solution.

1. Introduction

Several authors, Ericksen & Rivlin (1955), Oldroyd (1958), Noll (1958), Green & Rivlin (1960), have introduced properly invariant, kinematic theories of the stress tensor in non-Newtonian fluids. For one-dimensional, steady-state flows these theories describe the experimentally observed behaviour of non-Newtonian fluids, including both the shear-stress-strain-rate relationships and the normal-stress effect.

In the theory of Rivlin-Ericksen fluids (Ericksen & Rivlin 1955), the stress is assumed to be a function of the gradients of velocity, acceleration, second acceleration, . . . , $(n-1)$th acceleration. The matrix of components of the extra-stress tensor is given by

$$\mathbf{T} = f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n), \tag{1.1}$$

where $\mathbf{T}$ is defined by

$$\mathbf{T} = \mathbf{S} + P\mathbf{I}, \tag{1.2}$$

in which S is the stress tensor and P a scalar hydrostatic pressure. The matrices $\mathbf{A}_n$ represent the components of the Rivlin-Ericksen tensors, of which $\mathbf{A}_1$ is the familiar rate-of-strain tensor with components

$$a_{ij}^{(1)} = v_{i,j} + v_{j,i} \quad (i, j = 1, 2, 3). \tag{1.3}$$

Higher-order A_n are constructed from the recurrence relation

$$a_{ij}^{(n)} = \partial a_{ij}^{(n-1)}/\partial t + v^k a_{ij,k}^{(n-1)} + a_{ik}^{(n-1)} v^k_{,j} + a_{kj}^{(n-1)} v^k_{,i} \quad (i, j = 1, 2, 3;\ n > 1). \tag{1.4}$$

The constitutive relation f is usually represented as a polynomial in the A_n.

In a creeping motion the velocity v^i differs from rest by a small disturbance δv^i. If the A_n are evaluated using δv^i, the polynomial representation of equation (1.1) can be approximated to terms of order δv^i by the theory of Newtonian fluids. If terms of order $(\delta v^i)^2$ and $(\delta v^i)^3$ respectively are retained, the following stress equations are found for the incompressible case

$$\mathbf{T} = \phi_1 \mathbf{A}_1 + \phi_2 \mathbf{A}_2 + \phi_3 (\mathbf{A}_1)^2 + O(\delta v^i)^2, \tag{1.5}$$

$$\mathbf{T} = [\phi_1 + \phi_6 \operatorname{tr} (\mathbf{A}_1)^2] \mathbf{A}_1 + \phi_2 \mathbf{A}_2 + \phi_3 (\mathbf{A}_1)^2 + \phi_4 \mathbf{A}_3 + \phi_5 (\mathbf{A}_1 \mathbf{A}_2 + \text{A, A,}) + O(\delta v^i)^3, \tag{1.6}$$

where the ϕ_i are constants under isothermal conditions.

The theories expounded by Noll (1958) and Green & Rivlin (1960) include memory effects in which the stress state of the fluid at time t is assumed to depend on its kinematic state not only at time t but at all times in the past. Under the assumptions that Noll's 'simple fluid' is in slow motion and has a rapidly fading memory, Coleman & Noll (1961) have developed an approximation scheme which gives stress equations identical to those obtained above for the Rivlin-Ericksen fluids. Green & Rivlin (1960) have shown how their very general memory-theories can be specialized to give Noll's theory of simple fluids and also the theory of Rivlin-Ericksen fluids. Hence sound theoretical reasons exist for using stress equations (1.5) and (1.6) in the solution of problems of creeping flow of non-Newtonian fluids.

In a steady-state simple-shearing flow, a non-Newtonian fluid may be characterized by three material functions τ, σ_1 and σ_2. With the X^2-axis parallel to the stream lines and the Xl-axis normal to the shear planes the material functions are given by

$$\left.\begin{aligned} &\tau(\kappa) = S_{12}, && \text{the shear-stress function,} \\ &\sigma_1(\kappa) = S_{11} - S_{33} \text{ and } \sigma_2(\kappa) = S_{22} - S_{33}, && \text{the normal-stress functions,} \end{aligned}\right\} \tag{1.7}$$

all of which are determined by the rate of strain, κ, alone. If the stress equation (1.6) is evaluated for such a flow the following ϕ_i may be obtained from the material functions:

$$\left.\begin{aligned} \phi_1 &= \lim_{\kappa \to 0} \frac{\tau(\kappa)}{\kappa} = \eta_0, \quad \text{the zero-shear viscosity,} \\ (\phi_5 + \phi_6) &= \lim_{\kappa^2 \to 0} \frac{\{\tau(\kappa)/\kappa\} - \eta_0}{2\kappa^2}, \\ \text{and} \quad \phi_2 &= \lim_{\kappa^2 \to 0} \frac{\sigma_1(\kappa) - \sigma_2(\kappa)}{2\kappa^2}, \quad \phi_3 = \lim_{\kappa^2 \to 0} \frac{\sigma_2(\kappa)}{\kappa^2}, \end{aligned}\right\} \tag{1.8}$$

the zero-shear normal-stress coefficients. Most normal-stress measurements reported to date (Philippoff 1961) indicate the existence in simple shear of the symmetry relation $S_{11} = S_{33}$ which from (1.7) allows us to write

$$\phi_2 = -\tfrac{1}{2}\phi_3. \tag{1.9}$$

In addition, the molecular network theory of Lodge (1956) supports normal-stress symmetry, at least in the region of small rates of strain which is the domain of the stress equations (1.5) and (1.6).

2. The equations of motion

The equations of motion for a creeping flow are usually obtained simply by neglecting the inertial terms in the general equations of motion. In this way, and using the stress equations (1.6), Langlois & Rivlin (1959) have found approximate solutions to several flows contained within bounding surfaces. For creeping flows exterior to a body it is well known from Newtonian theory that near the uniform stream the inertial and viscous forces are of the same order. Uniform expansions to the stream function can be obtained by considering the Stokes expansions satisfying the no-slip condition at the surface of the body, and Oseen expansions satisfying the uniform-stream condition. Proudman & Pearson (1957) have shown how the two expansions can be matched to give a uniform approximation to the flow.

For the case of the sphere, polar co-ordinates (R, θ, ϕ) are chosen with the origin at the centre of the sphere and $\theta = 0$ in the upstream direction. It is convenient to work with $\mu = -\cos\theta$ instead of θ. All tensor quantities are expressed in terms of their physical components referred to these co-ordinates. The equations of motion, continuity and stress are first expressed in terms of the Stokes variables defined by

$$R = ar, \quad U_r = Uu_r, \quad U_\mu = Uu_\mu, \quad T_{ik} = (\eta_0 U/a)\, t_{ik}, \quad P = (\eta_0 U/a)\, p, \tag{2.1}$$

where a is the radius of the sphere, $\eta_0 = \phi_1$ the zero-shear viscosity, and U is the velocity of the uniform stream.

The stream function ψ is defined to satisfy the continuity equation

$$\frac{\partial(r^2 u_r)}{\partial r} + r\frac{\partial\{(1-\mu^2)^{\frac{1}{2}} u_\mu\}}{\partial\mu} = 0, \tag{2.2}$$

with

$$u_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, \quad \text{and} \quad u_\mu = \frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial r}. \tag{2.3}$$

If the pressure is eliminated from the steady-state equations of motion and the velocities are expressed in terms of ψ the following equation is obtained

$$\begin{aligned}\frac{1}{r^2}\frac{\partial(\psi, D^2\psi)}{\partial(r,\mu)} + \frac{2D^2\psi L\psi}{r^2} &= \frac{1}{Re}D^4\psi + \frac{1}{Re}\left[\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}[r^3(1-\mu^2)^{\frac{1}{2}}\tau_{r\mu}] + (1-\mu^2)^{\frac{1}{2}}\frac{\partial}{\partial\mu}[(1-\mu^2)^{\frac{1}{2}}\tau_{\mu\mu}] + \mu\tau_{\phi\phi}\right\}\right.\\ &\quad \left. -\frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{\partial}{\partial\mu}[(1-\mu^2)^{\frac{1}{2}}\tau_{r\mu}] - \tau_{\mu\mu} - \tau_{\phi\phi}\right\}\right],\end{aligned} \tag{2.4}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\mu^2)}{r^2}\frac{\partial^2}{\partial\mu^2}, \quad L = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\mu}, \tag{2.5}$$

and $Re = Ua/\nu_0$, the Reynolds number based on the zero-shear kinematic viscosity ν_0. The stress components $\tau_{r\mu}$, $\tau_{\mu\mu}$, etc., represent the non-Newtonian part of the stress matrix, which from (1.6) may be written in the new variables as

$$\mathbf{T} = \lambda(\mathbf{A}_1^2 + \epsilon_1\mathbf{A}_2) + \lambda^2[\epsilon_2 \operatorname{tr}(\mathbf{A}_1)^2\mathbf{A}_1 + \epsilon_3\mathbf{A}_3 + \epsilon_4(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1)], \tag{2.6}$$

where

$$\lambda = \phi_3 U/\eta_0 a, \quad \epsilon_1 = \phi_2/\phi_3, \quad \epsilon_2 = \phi_6\eta_0/\phi_3^2, \quad \epsilon_3 = \phi_4\eta_0/\phi_3^2, \quad \epsilon_4 = \phi_5\eta_0/\phi_3^2. \tag{2.7}$$

For small Reynolds numbers, the stream function ψ is assumed to have an inner (Stokes) expansion

$$\psi(r,\mu) = \psi_0(r,\mu) + f_1(Re)\,\psi_1(r,\mu) + \ldots, \tag{2.8}$$

where

$$f_{n+1}(Re)/f_n(Re) \to 0 \quad \text{as} \quad Re \to 0. \tag{2.9}$$

The Stokes expansion (2.8) is made to satisfy the governing equation (2.4), which gives for ψ_0

$$O = D^4\psi_0 + \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^3[1-\mu^2]^{\frac{1}{2}}\tau^0_{r\mu}) + [1-\mu^2]^{\frac{1}{2}}\frac{\partial}{\partial\mu}([1-\mu^2]^{\frac{1}{2}}\tau^0_{\mu\mu}) + \mu\tau^0_{\phi\phi}\right\}$$
$$-\frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r^2\tau^0_{rr}) + \frac{\partial}{\partial\mu}([1-\mu^2]^{\frac{1}{2}}\tau^0_{r\mu}) - \tau^0_{\mu\mu} - \tau^0_{\phi\phi}\right\}, \tag{2.10}$$

where $\tau^0_{r\mu}$, etc., are evaluated from (2.6) in terms of ψ_0. The solution of equation (2.10) is the chief object of this paper, and will be attempted in § 3.

Oseen variables are defined by

$$\rho = (Re)\,r, \text{ and } \Psi = (Re)^2\,\psi, \tag{2.11}$$

in terms of which the governing equation (2.4) becomes

$$\frac{1}{\rho^2}\frac{\partial(\Psi, D^2\Psi)}{\partial(\rho,\mu)} + \frac{2D^2\Psi L\Psi}{\rho^2} = D^4\Psi + Re(\text{non-Newtonian terms}), \tag{2.12}$$

where D^2 and L are the operators defined in (2.5), but with r replaced by ρ. The stream function, Ψ, in the outer region is assumed to have an Oseen expansion

$$\Psi = \Psi_0(\rho,\mu) + F_1(Re)\,\Psi_1(\rho,\mu) + F_2(Re)\,\Psi_2(\rho,\mu) + \ldots, \tag{2.13}$$

where

$$F_{n+1}(Re)/F_n(Re) \to 0 \quad \text{as} \quad Re \to 0. \tag{2.14}$$

The nature of the Oseen variables (2.11) is such that $\rho \to 0$ as $Re \to 0$, so that in the limit (2.12) describes the flow about a sphere of vanishing radius. Therefore, Ψ_0 must be the uniform stream

$$\Psi_0(\rho,\mu) = \tfrac{1}{2}\rho^2(1-\mu^2). \tag{2.15}$$

For the moment, F_1 is chosen as $F_1(Re) = Re$, with the provision that the unknown coefficients in the solution for Ψ_1 may be functions of Re. The Oseen expansion is now made to satisfy (2.12), which gives for Ψ_1

$$\frac{1-\mu^2}{\rho}\frac{\partial D^2\Psi_1}{\partial\mu} + \mu\frac{\partial D^2\Psi_1}{\partial\rho} = D^4\Psi_1. \tag{2.16}$$

But this is Oseen's equation for a Newtonian fluid with viscosity η_0; thus the non-linear terms do not influence the flow at large distances from the sphere. This result is compatible with the tendency of non-Newtonian fluids toward Newtonian behaviour at low rates of strain.

3. The leading term in the Stokes expansion

The solution of equation (2.10) is the leading term ψ_0 in the Stokes expansion. It must satisfy the no-slip condition on the sphere and match with the leading term in the Oseen expansion, the uniform stream Ψ_0, (2.15). The matching procedure is based on the idea that ψ and Ψ are different forms of the same function. This requires that ψ_n, expressed in terms of the Oseen variable ρ, should match with the Ψ_n for small values of *Re*. The details of this procedure have been illustrated very clearly by Proudman & Pearson, and will not be discussed further.

The non-Newtonian stress matrix (2.6) suggests for ψ_0 the form

$$\psi_0 = \chi_{00} + \lambda\chi_{01} + \lambda^2\chi_{02} + \ldots, \tag{3.1}$$

where $\lambda = \phi_3 U/\eta_0 a < 1$, and the ϵ_i are assumed to be of order one.* The expansion (3.1) allows the non-Newtonian stress appearing in (2.10) to have a matrix representation

$$\mathbf{T}^0 = \lambda\mathbf{T}^{01} + \lambda^2\mathbf{T}^{02} + \ldots, \tag{3.2}$$

with components $\tau^0_{rr} = \lambda\tau^{01}_{rr} + \lambda^2\tau^{02}_{rr} + \ldots$, etc. The expansions (3.1) and (3.2) are substituted into (2.10) and the coefficients of each power of λ are equated, which gives

$$D^4\chi_{00} = 0, \tag{3.3}$$

$$\begin{aligned} 0 = D^4\chi_{01} &+ \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^3[1-\mu^2]^{\frac{1}{2}}\tau^{01}_{r\mu}) + [1-\mu^2]^{\frac{1}{2}}\frac{\partial}{\partial\mu}([1-\mu^2]^{\frac{1}{2}}\tau^{01}_{\mu\mu}) + \mu\tau^{01}_{\phi\phi}\right\} \\ &- \frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r^2\tau^{01}_{rr}) + \frac{\partial}{\partial\mu}([1-\mu^2]^{\frac{1}{2}}\tau^{01}_{r\mu}) - \tau^{01}_{\mu\mu} - \tau^{01}_{\phi\phi}\right\}, \quad \text{etc.} \end{aligned} \tag{3.4}$$

The solution of equation (3.3) is chosen to satisfy the no-slip condition at the surface and is matched with the uniform stream (2.15). Clearly χ_{00} must be the Stokes solution (Lamb 1932, p. 598) for the Newtonian case, i.e.

$$\chi_{00} = \tfrac{1}{2}(r^2 - \tfrac{3}{2}r + \tfrac{1}{2}r^{-1})(1-\mu^2). \tag{3.5}$$

From (2.6) the matrix of components of the non-Newtonian stress appearing in (3.4) is given by

$$\mathbf{T}^{01} = (\mathbf{A}^{00}_1)^2 + \epsilon_1\mathbf{A}^{00}_2, \tag{3.6}$$

where $\mathbf{A}^{00}_1$ and $\mathbf{A}^{00}_2$ are evaluated in terms of χ_{00}, which gives for the components of T^{01}

$$\left.\begin{aligned} \tau^{01}_{rr} &= \frac{3\mu^2}{r^3}\left[\frac{3}{r}\left(1-\frac{1}{r^2}\right)^2 + 2\epsilon_1\left(-1+\frac{3}{r}+\frac{2}{r^2}-\frac{13}{r^3}+\frac{5}{2r^5}\right)\right] \\ &\qquad + \frac{3(1-\mu^2)}{r^3}\left[\frac{3}{4r^5} + 2\epsilon_1\left(\frac{1}{2}-\frac{3}{8r}-\frac{1}{r^2}+\frac{1}{r^3}+\frac{5}{8r^5}\right)\right], \\ \tau^{01}_{\mu\mu} &= \frac{3\mu^2}{r^3}\left[\frac{3}{4r}\left(1-\frac{1}{r^2}\right)^2 + 2\epsilon_1\left(\frac{1}{2}-\frac{3}{8r}-\frac{1}{r^2}+\frac{1}{r^3}-\frac{1}{8r^5}\right)\right] \\ &\qquad + \frac{3}{4}\frac{(1-\mu^2)}{r^3}\left[\frac{3}{r^5} + 2\epsilon_1\left(-1+\frac{3}{4r}+\frac{3}{r^2}-\frac{7}{2r^3}+\frac{3}{4r^5}\right)\right], \\ \tau^{01}_{\phi\phi} &= \frac{3\mu^2}{r^3}\left[\frac{3}{4}\left(1-\frac{1}{r^2}\right)^2 + 2\epsilon_1\left(\frac{1}{2}-\frac{3}{8r}-\frac{1}{r^2}+\frac{1}{r^3}-\frac{1}{8r^5}\right)\right] \\ &\qquad + \frac{3}{4}\frac{(1-\mu^2)}{r^3}\left[2\epsilon_1\left(-1+\frac{3}{4r}+\frac{1}{r^2}-\frac{1}{2r^3}-\frac{1}{4r^5}\right)\right], \\ \tau^{01}_{r\mu} &= \frac{3\mu\sqrt{(1-\mu^2)}}{r^3}\left[+\frac{3}{4r^3}\left(1-\frac{1}{r^2}\right) - 2\epsilon_1\left(-\frac{3}{4}+\frac{9}{8r}+\frac{2}{r^2}-\frac{3}{r^3}+\frac{5}{8r^5}\right)\right]. \end{aligned}\right\} \tag{3.7}$$

* This assumption is partially justified in the Appendix.

Hence (3.4) becomes

$$D^4\chi_{01} = -27(1+\epsilon_1)\left(1-\frac{2}{r^2}\right)\frac{\mu(1-\mu^2)}{r^5}. \tag{3.8}$$

Because the Newtonian solution, χ_{00}, has already been fully matched with the free stream, the non-Newtonian terms $\chi_{01}, \chi_{02}, \ldots$, in ψ_0 must not contribute to Ψ_0. This also satisfies the requirement of Oseen's equation for Newtonian behaviour near the free stream. A solution to (3.8) satisfying the no-slip condition at $r = 1$, and which does not contribute to Ψ_0 is

$$\chi_{01} = -\tfrac{3}{8}(1+\epsilon_1)\left(1-\frac{1}{r}\right)^3\mu(1-\mu^2). \tag{3.9}$$

I	II	III	IV	V	VI
b_6	14	35	0	−7	−54
b_7	−174	−216	54	21/2	475/4
b_8	370	400	0	18	201/4
b_9	−106	−174	−168	−61/2	−869/4
b_{10}	−250	−165	0	0	76
b_{11}	144	120	60	9	93/4
b_{12}	0	0	0	0	35/4
b_{13}	0	0	68	0	13
c_6	−16	−40	0	8	243/4
c_7	198	243	−54	−45/4	−126
c_8	−420	−885/2	0	−20	−285/4
c_9	167	204	165	127/4	1943/8
c_{10}	162	138	0	0	−459/8
c_{11}	−87	−102	−42	−17/2	−54
c_{12}	0	0	0	0	−5/2
c_{13}	0	0	−30	0	−45/4

TABLE 1

I = (27/4) $(1+\epsilon_1)$ (II) + (27/2) $\epsilon_1(1+\epsilon_1)$ (III) + (27/2) $(\epsilon_2+\epsilon_4)$ (IV) + $27\epsilon_4$(V) + $6\epsilon_3$(VI)

The equation for χ_{02} is analogous to (3.4), and contains the components τ^{02}_{rr}, etc., of the non-Newtonian stress matrix

$$\mathbf{T}^{02} = \mathbf{A}_1^{01}\mathbf{A}_1^{00} + \mathbf{A}_1^{00}\mathbf{A}_1^{01} + \epsilon_1\mathbf{A}_2^{01,00} + \epsilon_2\,\mathrm{tr}\,(\mathbf{A}_1^{00})^2\,\mathbf{A}_1^{00} + \epsilon_3\mathbf{A}_3^{00} + \epsilon_4(\mathbf{A}_1^{00}\mathbf{A}_2^{00} + \mathbf{A}_2^{00}\mathbf{A}_1^{00}), \tag{3.10}$$

where $\mathbf{A}_1^{01}$, $\mathbf{A}_1^{00}$, $\mathbf{A}_2^{00}$, etc., are evaluated in terms of χ_{00} and χ_{01}. The resulting differential equation for χ_{02} is

$$D^4\chi_{02} = \sum_{n=6}^{13}[b_n + c_n(1-\mu^2)]\,r^{-n}(1-\mu^2), \tag{3.11}$$

where the coefficients b_n and c_n are given in table 1. A solution to (3.11) satisfying the no-slip condition at $r = 1$, and which does not contribute to the free stream, is

$$\chi_{02} = \left\{[(1-\mu^2)-\tfrac{4}{5}]\,\delta r^{-3}\ln r + \sum_{\substack{m=-1\\ m\neq 0}}^{9}[\beta_m + \gamma_m(1-\mu^2)]\,r^{-m}\right\}(1-\mu^2), \tag{3.12}$$

where δ, β_m, and γ_m are shown in table 2.

I	II	III	IV	V	VI
δ	−1·5714	−1·9286	0·4286	0·0893	1·0000
β_{-1}	0·2419	0·2983	−0·0431	−0·0138	−0·4570
β_1	−0·5148	−0·5568	0·1171	0·0049	1·5364
β_2	−0·2500	−0·6250	0	0·1250	−0·5812
β_3	−0·6948	−0·3932	0·0947	−0·1526	−0·5129
β_4	1·4472	1·5389	0	0·0694	0·2281
β_5	−0·1504	−0·2132	−0·1905	−0·0354	−0·2605
β_6	−0·1150	−0·0808	0	0	0·0362
β_7	0·0359	0·0319	0·0153	0·0024	0·0084
β_8	0	0	0	0	0·0012
β_9	0	0	0·0064	0	0·0013
γ_{-1}	0	0	0	0	0
γ_1	0·0392	−0·0611	−0·0400	0·0321	−0·3624
γ_2	0·3333	0·8333	0	−0·1667	−0·1266
γ_3	1·1019	0·7670	−0·1528	0·1802	0·5285
γ_4	−1·7500	−1·8437	0	−0·0833	−0·2969
γ_5	0·2109	0·2576	0·2083	0·0401	0·3067
γ_6	0·0900	0·0767	0	0	−0·0319
γ_7	−0·0253	−0·0297	−0·0122	−0·0025	−0·0157
γ_8	0	0	0	0	−0·0004
γ_9	0	0	−0·0032	0	−0·0012

TABLE 2

$\mathrm{I} = (27/4)\,(1+\epsilon_1)\,(\mathrm{II}) + (27/2)\,(1+\epsilon_1)\,\epsilon_1(\mathrm{III}) + (27/2)\,(\epsilon_2+\epsilon_4)\,(\mathrm{IV}) + 27\epsilon_4(\mathrm{V}) + 6\epsilon_3(\mathrm{VI})$

4. Higher terms in the Oseen and Stokes expansions

Since the matching procedure for the inner and outer expansions involves only the Newtonian terms, the solution to Oseen's equation (2.16) will be that given by Proudman & Pearson

$$\Psi_1 = -\tfrac{3}{2}(1+\mu)\,[1-\exp\{-\tfrac{1}{2}\rho(1-\mu)\}]. \tag{4.1}$$

The coefficient f_1 in the Stokes expansion is set as $f_1(Re) = Re$ with the same provision made on F_1 in § 2. The governing equation (2.4) yields, for ψ_1,

$$D^4\psi_1 + O(\lambda) = \frac{9}{2}\left(-\frac{1}{r^2}+\frac{3}{2r^3}+\frac{1}{2r^5}\right)\mu(1-\mu^2)+O(\lambda). \tag{4.2}$$

The terms of $O(\lambda)$ on the left are evaluated from the non-Newtonian stress matrix (2.4) using $\psi_0 = \chi_{00}+\lambda\chi_{01}$, and ψ_1, and hence will be linear in the derivatives of ψ_1 with coefficients which are functions of r and μ. The terms on the right of (4.2) are evaluated from the inertial terms of (2.6) using ψ_0. The first term is the Newtonian contribution, and the terms of $O(\lambda)$ are obtained from χ_{01}.

In this paper terms of $O(\lambda)$ in ψ_1 will be neglected by imposing the restriction

$$Re \leqslant \lambda^2 \ll \lambda, \tag{4.3}$$

which is equivalent to

$$a^2 \ll \phi_3/d, \tag{4.4}$$

where d is the density of the fluid. Condition (4.4) is easily achieved experimentally by choosing a small enough sphere.

Under the above restriction, (4.2) reduces to the Newtonian case. Its solution, properly matched with Ψ_1, is given by Proudman & Pearson as

$$\chi_{10} = \frac{3}{32}\left(2r^2-3r+\frac{1}{r}\right)(1-\mu^2)-\frac{3}{32}\left(2r^2-3r+1-\frac{1}{r}+\frac{1}{r^2}\right)(1-\mu^2), \tag{4.5}$$

where ψ_1 has been put in the form $\psi_1 = \chi_{10}+\lambda\chi_{11}+\ldots$, to conform with (3.1).

5. The stream function

The stream function has now been determined in the form

$$\psi = \chi_{00}+\lambda\chi_{01}+\lambda^2\chi_{02}+O(\lambda^3)+Re\chi_{10}+O(\lambda Re), \tag{5.1}$$

or from (3.5), (3.9), (3.12), and (4.5)

$$\psi = (1-\mu^2)\Bigg[\tfrac{1}{4}(r-1)^2\Bigg\{(1+\tfrac{3}{8}Re)\left(2+\frac{1}{r}\right)-\tfrac{3}{8}Re\left(2+\frac{1}{r}+\frac{1}{r^2}\right)\mu - \frac{3\lambda}{2r^3}(1+\epsilon_1)(r-1)\mu\Bigg\}+\lambda^2\Bigg\{[(1-\mu^2)-\tfrac{4}{5}]\,\delta r^{-3}\ln r + \sum_{\substack{m=-1\\ m\neq 0}}^{9}[\beta_m+\gamma_m(1-\mu^2)]\,r^{-m}\Bigg\}\Bigg]. \tag{5.2}$$

An interesting property of the flow is the possibility of the formation of eddies behind the sphere. The stream function ψ has zeros at $\mu = \pm 1$ and on the surface $r = 1$; in addition it vanishes along the real curve whose polar equation is

$$\mu = \frac{(\frac{8}{3}Re+1)(2r^3+r^2)}{2r^3+r^2+r+(4\lambda/Re)(1+\epsilon_1)(r-1)}, \tag{5.3}$$

and this curve forms the boundary of the eddies. The term in λ^2 was dropped from (5.2) to obtain (5.3). For $\lambda < 1$ this will be a small correction so that (5.3) is sufficient to determine the existence of eddies. The minimum of (5.3) occurs when

$$4(\alpha+1)r^2+(1-5\alpha)r-2\alpha = 0, \tag{5.4}$$

where $\alpha = (4\lambda/Re)(\epsilon_1+1)$.

Two cases are considered.

(1) $\lambda = Re$: by (1.9) $\epsilon_1 = -\frac{1}{2}$ and $\alpha = 2$, from (5.4) $r = 1{\cdot}06$, hence

$$\mu_{\min} \approx (2/Re)+\tfrac{3}{4} \quad \text{which means} \quad \lambda = Re \approx 8.$$

(2) $\lambda = 10Re$: $\alpha = 20$, $r = 1{\cdot}5$, $\mu_{\min} = (1{\cdot}16/Re)+0{\cdot}44$, hence $Re = 2{\cdot}1$, $\lambda = 21$.

Cases 1 and 2 show that eddies can form only at values of Re and λ which are well beyond the theory of slow motions and slight deviations from Newtonian behaviour from which ψ was obtained. Therefore, it is not possible to claim that equation (5.2) gives even a qualitative description of the flow for such large values of these parameters. Leslie (1961) has investigated the flow around a sphere using the Oldroyd (1958) model, and has obtained a solution by expanding in terms of a parameter similar to λ. His first correction to the Newtonian solution is identical to χ_{01} and the second is qualitatively similar to χ_{02}. Leslie has plotted these functions for reasonable values of the parameters of Oldroyd's model, and finds the stream function differs only slightly from the Stokes solution with no eddies appearing behind the sphere.

6. The drag force on the sphere

The fluid will exert a drag force on the sphere which can be computed from

$$D = 2\pi a\eta_0 U \int_{-1}^{1} [(S_{rr})_{r=1}\mu + (S_{r\mu})_{r=1}(1-\mu^2)^{\frac{1}{2}}]\,d\mu, \tag{6.1}$$

which may be expressed in the notation of (5.1) as

$$D/2\pi a\eta_0 U = d_{00} + \lambda d_{01} + \lambda^2 d_{02} + \ldots + Re\, d_{10} + \lambda Re\, d_{11} + \ldots, \tag{6.2}$$

where the d_{ij} are constants given by

$$d_{ij} = \int_{-1}^{1} [(S_{rr}^{ij})_{r=1}\mu + (S_{r\mu}^{ij})_{r=1}(1-\mu^2)^{\frac{1}{2}}]\,d\mu, \tag{6.3}$$

in which S_{rr}^{ij}, $S_{r\mu}^{ij}$ are components of the stress tensor (1.2) expanded in similar fashion to (3.2). For $i = 0$, this gives $d_{00} = 3$, $d_{01} = 0$ and

$$d_{02} = 13{\cdot}5[4{\cdot}5278(1+2{\cdot}4528\epsilon_1)(1+\epsilon_1) + 3{\cdot}0984(\epsilon_2+\epsilon_4) - 1{\cdot}0296\epsilon_4 - 5{\cdot}6160\epsilon_3], \tag{6.4}$$

for $i = 1$,

$$d_{10} = \tfrac{9}{8}. \tag{6.5}$$

The final expression for the drag is

$$D = 6\pi a\eta_0 U(1+\tfrac{3}{8}Re) + \frac{27\pi U^3}{a}\left[4{\cdot}5278\left(1+2{\cdot}4528\frac{\phi_2}{\phi_3}\right)\left(1+\frac{\phi_2}{\phi_3}\right)\frac{\phi_3^2}{\eta_0} + 3{\cdot}0984(\phi_5+\phi_6) - 5{\cdot}6160\phi_4 - 1{\cdot}0296\phi_5\right], \tag{6.6}$$

which is Stokes law with a small Reynolds-number correction and a term in U^3 whose coefficient involves the non-Newtonian parameters, of which ϕ_4 and ϕ_5 cannot be obtained from simple shear flows, and so the magnitude of the term cannot be estimated from available experimental data.

The drag expression (6.6) is complete in the sense that the non-Newtonian parameters of the higher approximations to the stress equation (1.6) do not affect the drag until terms in λ^3 are considered. This makes it an experimentally useful formula.

Leslie computed the drag from his solution for the Oldroyd model, and obtained a non-Newtonian term proportional to U^3. Hence slow-flow drag-force measurements may be interpreted using either the Oldroyd or Rivlin–Ericksen models with equal justification, a state of affairs which is not too surprising since both models predict Newtonian behaviour in the limit of very slow flow. The domain of the solutions presented here and in Leslie's work covers the region only slightly beyond the Newtonian limit; therefore, it is not unreasonable that both models should approach this limit in similar fashion.

Appendix

The data of Philippoff (1956) on a 15 % solution of polyisobutylene (B-100) in decalin at 30 °C has been analysed using the relationships (1.8) between material functions and the constants ϕ_i. This gives

$$\phi_1 = \eta_0 = 9320\,\text{poise},$$
$$\phi_3 = 45{,}000\,\text{dyn}\,\text{cm}^{-2}\,\text{sec}^2 \quad (\text{assuming } \phi_2 = -\tfrac{1}{2}\phi_3),$$
$$\phi_5 + \phi_6 = 22{,}000\,\text{dyn}\,\text{cm}^{-2}\,\text{sec}^3;$$

hence $$\epsilon_2 + \epsilon_4 = (\phi_5 + \phi_6)\,\eta_0/\phi_3^2 = 0{\cdot}10.$$

The coefficient ϕ_4 cannot be estimated from simple shear data, and ϕ_5 and ϕ_6 cannot be obtained individually.

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